NEWTON–GREGORY INTERPOLATING POLYNOMIALS

Difference Table

	i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$	$\Delta^5 f_i$
	0	10	6	$+10$	-4	$+1$	-2	$+4$
	1	15	16	$+6$	-3	-1	$+2$	
	2	20	22	$+3$	-4	$+1$		
$h=5$	3	25	25	-1	-3			
	4	30	24	-4				
	5	35	20					

1. Let's interpolate $f(16)$ using polynomials of different degrees, each beginning with abscissa $x_0 = 10$. So $s = \frac{x-x_0}{h} = \frac{16-10}{5} = 1.2$. Determine the binomial coefficients via its telescopic property:

$$\begin{aligned}
\binom{s}{1} &= s = +1.2\,, \\
\binom{s}{2} &= \binom{s}{1}\cdot\frac{s-1}{2} = +1.2\cdot\left(\frac{1.2-1}{2}\right) = +0.12\,, \\
\binom{s}{3} &= \binom{s}{2}\cdot\frac{s-2}{3} = +0.12\cdot\left(\frac{1.2-2}{3}\right) = -0.032\,, \\
\binom{s}{4} &= \binom{s}{3}\cdot\frac{s-3}{4} = -0.032\cdot\left(\frac{1.2-3}{4}\right) = +0.0144\,, \\
\binom{s}{5} &= \binom{s}{4}\cdot\frac{s-4}{5} = +0.0144\cdot\left(\frac{1.2-4}{5}\right) = -0.008064\,,
\end{aligned}$$

   (a) Let's use the 2nd degree interpolating polynomial containing points [0,1,2]:

$$P_{[0-2]}(x) = f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0\,, \qquad 10 \le x \le 20\,, \tag{1}$$

$$\begin{aligned}
f(16) &\approx P_{[0-2]}(s=1.2) \\
&= 6 + (1.2)\cdot(10) + (0.12)\cdot(-4) = \boxed{17.52}\,.
\end{aligned} \tag{2}$$

   (b) Let's use the 3rd degree interpolating polynomial containing points [0,1,2,3]:

$$P_{[0-3]}(x) = \underbrace{f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0}_{P_{[0-2]}(x)} + \underbrace{\binom{s}{3}\Delta^3 f_0}_{\text{“next term”}}\,, \qquad 10 \le x \le 25\,. \tag{3}$$

   So

$$\begin{aligned}
f(16) &\approx P_{[0-3]}(s=1.2) \\
&= P_{[0-2]}(s=1.2) + \binom{s}{3}\Delta^3 f_0 \\
&= 17.52 + (-0.032)\cdot(+1)\,, \qquad \text{by Result (2)} \\
&= 17.52 - 0.032 \qquad (4) \\
&= \boxed{17.488}\,. \qquad (5)
\end{aligned}$$

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(c) Similarly, if we use the 4th degree interpolating polynomial containing points [0,1,2,3,4]:

$$\begin{aligned} f(16) &\approx P_{[0-4]}(s=1.2)\,, \qquad 10 \le x \le 30 \\ &= P_{[0-3]}(s=1.2) \;+\; \binom{s}{4}\Delta^4 f_0 \\ &= 17.488 \;+\; (0.0144)\cdot(-2) \qquad \text{by Result (5)} \\ &= 17.488 \;-\; 0.0288 \\ &= \boxed{17.4592}\,. \end{aligned} \tag{6}$$

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2. Let's interpolate $f(16)$ using polynomials of different degrees, each beginning with abscissa $x_1 = 15$. So $s = \frac{x-x_1}{h} = \frac{16-15}{5} = 0.2$. Let's determine the binomial coefficients:

$$\begin{aligned} \binom{s}{1} &= s \;=\; 0.2\,, \\ \binom{s}{2} &= \binom{s}{1}\cdot\frac{s-1}{2} \;=\; 0.2\cdot\left(\frac{0.2-1}{2}\right) \;=\; -0.08\,, \\ \binom{s}{3} &= \binom{s}{2}\cdot\frac{s-2}{3} \;=\; -0.08\cdot\left(\frac{0.2-2}{3}\right) \;=\; 0.048\,, \\ \binom{s}{4} &= \binom{s}{3}\cdot\frac{s-3}{4} \;=\; 0.0488\cdot\left(\frac{0.2-3}{4}\right) \;=\; -0.0336\,. \end{aligned}$$

(a) Let's use the 2nd degree interpolating polynomial containing points [1,2,3]:

$$P_{[1-3]}(x) \;=\; f_1 \;+\; \binom{s}{1}\Delta f_1 \;+\; \binom{s}{2}\Delta^2 f_1\,, \qquad 15 \le x \le 25\,, \tag{7}$$

$$\begin{aligned} f(16) &\approx P_{[1-3]}(s=0.2) \\ &= 16 \;+\; (0.2)\cdot(6) \;+\; (-0.08)\cdot(-3) \\ &= \boxed{17.44}\,. \end{aligned} \tag{8}$$

(b) Let's use the 3rd degree interpolating polynomial containing points [1,2,3,4]:

$$P_{[1-4]}(x) \;=\; P_{[1-3]}(x) \;+\; \binom{s}{3}\Delta^3 f_1\,. \qquad 15 \le x \le 30\,, \tag{9}$$

$$\begin{aligned} f(16) &\approx P_{[1-3]}(s=0.2) \;+\; \binom{s}{3}\Delta^3 f_1 \\ &= 17.44 \;+\; (0.048)\cdot(-1) \quad \text{by Result (8)} \\ &= 17.44 \;-\; 0.048 \\ &= \boxed{17.392}\,. \end{aligned} \tag{10}$$

(c) Let's use the 4th degree interpolating polynomial containing points [1,2,3,4,5]:

$$P_{[1-5]}(x) \;=\; P_{[1-4]}(x) \;+\; \binom{s}{4}\Delta^4 f_1\,. \qquad 15 \le x \le 35\,, \tag{11}$$

$$\begin{aligned} f(16) &\approx P_{[1-4]}(s=0.2) \;+\; \binom{s}{4}\Delta^4 f_1 \\ &= 17.392 \;+\; (-0.0336)\cdot(2) \quad \text{by Result (10)} \\ &= 17.392 \;-\; 0.0672 \\ &= \boxed{17.3248}\,. \end{aligned} \tag{12}$$

NEXT TERM RULE: The *next term* can be used to estimate the interpolation error.

1. To estimate the error in using $P_{[0-2]}(x)$ to interpolate at $x = 16$:

$$\text{Error} \;\approx\; \text{“next term”} \;=\; \binom{s}{3}\Delta^3 f_0 \;=\; (-0.032)(1) \;=\; -0.032\,.$$

   So using points $[0, 1, 2]$, $f(16) \approx 17.52$ with an error of about Error ≈ -0.032.

2. To estimate the error in using $P_{[0-3]}(x)$ to interpolate at $x = 16$:

$$\text{Error} \;\approx\; \text{“next term”} \;=\; \binom{s}{4}\Delta^4 f_0 \;=\; (0.0144)(-2) \;=\; -0.288\,.$$

   So using points $[0, 1, 2, 3]$, $f(16) \approx 17.488$ with an error of about Error ≈ -0.0288.

3. To estimate the error in using $P_{[1-3]}(x)$ to interpolate at $x = 16$:

$$\text{Error} \;\approx\; \text{“next term”} \;=\; \binom{s}{3}\Delta^3 f_1 \;=\; (0.048)(-1) \;=\; -0.048\,.$$

   So using points $[1, 2, 3]$, $f(16) \approx 17.44$ with an error of about Error ≈ -0.048.